Stability of spatially developing boundary layers

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Abstract: A new formulation of the stability of boundary-layer flows in pressure gradients is presented, taking into account the spatial development of the flow. The formulation assumes that disturbance wavelength and eigenfunction vary downstream no more rapidly than the boundary-layer thickness, and includes all terms of $O(1)$ and $O(R^{-1})$ in the boundary-layer Reynolds number R. Although containing the Orr-Sommerfeld operator, the present approach does not yield the Orr-Sommerfeld equation in any rational limit. In Blasius flow, the present stability equation is consistent with that of Bertolotti *et al.* (1992) to terms of $O(R^{-1})$. For the Falkner-Skan similarity solutions neutral boundaries are computed without the necessity of having to march in space. Results show that the effects of spatial growth are striking in flows subjected to adverse pressure gradients.

1. Introduction

A laminar boundary layer becomes unstable to small disturbances at some downstream distance beyond which the disturbances grow, undergo nonlinear interactions, and ultimately cause the transition of the boundary layer to turbulence. In the classical linear stability analysis (described for example in Drazin and Reid, 1981) of the flow over a flat plate, the disturbance is broken up into normal modes of the form $\phi(y)e^{i(\alpha x-\omega t)}$ where x is the streamwise distance, y is the distance perpendicular to the wall, t is time, and α and ω are the wave number and frequency of the disturbance, respectively. Only two-dimensional disturbances are considered here since, for a two-dimensional mean flow, they become unstable at a Reynolds number lower than that for three-dimensional disturbances (Squire's theorem). Since the disturbances are assumed small, their products may be neglected. If it is further assumed that the boundary layer is locally parallel, i.e., it does not grow with x (so $\partial/\partial x = 0$ and the normal velocity is zero), then ϕ satisfies the Orr-Sommerfeld equation

$$\mathcal{L}\phi = 0 \tag{1}$$

where $\mathcal{L}$ is the Orr-Sommerfeld operator given by

$$i\left(\omega - \alpha\phi_0'\right)D^2 + i\alpha\left(-\alpha(\omega - \alpha\phi_0') + \phi_0'''\right) + \frac{1}{R}\left(D^4 - 2\alpha^2 D^2 + \alpha^4\right).$$

This equation has been non-dimensionalised using the freestream velocity U, and the boundary-layer thickness δ. The Reynolds number based on these scales is R, $\phi_0(y)$ is the mean streamfunction, and D^k is the k-th order spatial derivative, i.e.,

$$D^k \equiv \frac{d^k}{dy^k}, \quad k = 1, 2, \ldots$$

In spatial stability analysis, α is taken to be complex and ω to be real, the boundary layer being unstable to a given disturbance if the imaginary part of its wave number is negative.

The curve for neutral stability obtained from Eq. (1) is shown in Fig. 1 (Gaster 1974; also present computations). Here, the non-dimensional frequency $F = \omega/R$ corresponding to neutral disturbances is plotted as a function of the Reynolds number $R = U\theta/\nu$, where θ and ν are the momentum thickness and kinematic viscosity, respectively. It is seen that it does not match experiment exactly. Several workers have investigated the stability of non-parallel flows in the expectation that this discrepancy may be explained in whole or in part as due to flow non-parallelism. However, the subject has been controversial. Gaster (1974) approached the problem assuming that the disturbances in developing flow could he described in terms of local eigenfunctions with x-dependent amplitudes by means of a perturbation of the Orr-Sommerfeld problem. He pointed out that the neutral stability boundary depends on the flow quantity considered (*e.g.*, streamwise disturbance velocity, disturbance kinetic energy, *etc.*). Fasel and Konzelmann (1990) have solved the complete Navier-Stokes equation numerically to obtain non-parallel effects, while Bertolotti *et al.* (1992) derived a parabolic stability equation for the flow over a flat plate assuming slow variations in the streamwise direction. The results of both these approaches agree with Gaster's. Sen (1992) has proposed a modified Orr-Sommerfeld equation that uses a disturbance streamfunction nondimensionalised by the boundary-layer thickness which is assumed to be a constant locally. His results agree better with experiment than those of other researchers. Smith (1979) used the triple-deck theory at large Reynolds numbers where the non-parallel effects emerge as higher order terms in the asymptotic expansion.

We study here the stability of the Falkner-Skan similarity solutions of the boundary-layer equations (see *e.g.*, Drazin and Reid 1981). An extension of the present ideas to non-similar basic flows will he published separately. Our approach assumes that the disturbance parameters (a, ϕ) vary slowly in *x*, *i.e.*, the derivatives in the *x* direction are $O(R^{-1})$ smaller than those in the similarity variable y, and contains all terms upto $O(R^{-1})$ for a similar boundary layer. The present equation is consistent with Eq. (6a) of Bertolotti *et al.* (1992) for a constant freestream velocity. A significant difference however is that we solve the equation here at one step, and not by space marching as done by Bertolotti *et al.* (1992). The method of solution is described in section 3. The boundary conditions used are of the form proposed by Sen (1992).

2. Formulation

In two-dimensional flow, the Navier-Stokes equations may be written in terms of the streamfunction as

$$\frac{\partial}{\partial t_d}\nabla_d^2\psi_d + \frac{\partial\psi_d}{\partial y_d}\frac{\partial}{\partial x_d}\nabla_d^2\psi_d - \frac{\partial\psi_d}{\partial x_d}\frac{\partial}{\partial y_d}\nabla_d^2\psi_d - \nu\nabla_d^4\psi_d = 0. \tag{2}$$

Here the subscript d stands for a dimensional quantity. The above equation is non-dimensionalised using

$$\psi = \frac{\psi_d}{(U\theta)}, \quad \theta(x_d)dx = dx_d \quad \text{and} \quad y = \frac{y_d}{\theta}$$

where $U(x_d)$ and $\theta(x_d)$ are the local dimensional freestream velocity and momentum thickness respectively. For a Falkner-Skan profile, $U \propto x_d^m$ where m is a constant. Therefore

$$x = \frac{2x_d}{(1+m)\theta}, \quad \frac{d\theta}{dx_d} = \frac{Uq}{R}, \quad \frac{d(U\theta)}{dx_d} = \frac{Up}{R},$$

where p and q are constants given by

$$p = \theta^{*2}, \quad q = \theta^{*2}\frac{(1-m)}{(1+m)}, \quad \theta^* \equiv \int \phi_0'(1-\phi_0')dy.$$

The streamfunction is now split into a mean and a fluctuating quantity:

$$\psi = \phi_0(y) + \phi(x,y)\exp\left(i\int \alpha(x)dx - \omega_d t_d\right).$$

We note that $d\theta/dx_d = O(R^{-1})$, and assume that a and ϕ cannot vary faster (in x) than θ *i.e.* their first derivatives with respect to x are at most of order $1/R$, and that their second derivatives are $o(R^{-1})$ and can therefore be neglected. On substituting the above expression for ψ in Eq. (2) it is seen that the mean streamfunction is given by

$$\phi_0^{iv} + p\phi_0\phi_0''' + (2q-p)\phi_0'\phi_0'' = 0 \tag{3}$$

which is the Falkner-Skan equation differentiated once with respect to y. The disturbance streamfunction is given by

$$\Big\{ i(\omega - \alpha\phi_0')D^2 + i\alpha\left[-\alpha(\omega - \alpha\phi_0') + \phi_0'''\right] + \frac{1}{R}\Big(D^4 + p\phi_0 D^3 + \left[-2\alpha^2 + (2q-p)\phi_0'\right]D^2$$
$$+ \left[2yq\alpha(\omega - \alpha\phi_0') - p\alpha^2\phi_0 + (2q-p)\phi_0''\right]D + \left[\alpha^4 + (q-2p)\alpha\omega + p\phi_0''' + 3(p-q)\alpha^2\phi_0'\right]$$
$$+ (-\omega + 3\alpha\phi_0')R\frac{d\alpha}{dx} + \left[\phi_0''' + 3\alpha^2\phi_0' - 2\alpha\omega - \phi_0'D^2\right]R\frac{\partial}{\partial x}\Big)\Big\}\phi = 0. \tag{4}$$

Here, terms of $O(y/R^2)$ have been neglected, hence the above equation is valid in the region where $y \sim o(R)$. Since $d^2\alpha/dx^2$ and $\partial^2\phi/\partial x^2$ are negligible, at a given Reynolds number $d\alpha/dx$ is a number and $\partial\phi/\partial x$ is a function of y. The above equation is therefore an ordinary differential equation in y. The boundary conditions are

$$\phi = D\phi = 0 \quad \text{at} \quad y = 0 \quad \text{and}$$

$$\phi \to 0,\ D\phi \to 0 \text{ as } y \to \infty.$$

For a finite computational domain, ϕ and $D\phi$ are decaying quantities at large y. At $y > \delta$, the Orr-Sommerfeld equation (Eq. 1) reduces to one with constant coefficients which can be solved to obtain $\phi \sim e^{-\alpha y}$. In the non-parallel formulation (Eq. 4), however, the coefficients of the first and third derivatives in y of ϕ are proportional to y outside the boundary layer. Therefore the asymptotic expression for ϕ when $y \gg 6$ is of the form found by Sen (1992) for his modified Orr-Sommerfeld equation, and may be worked out from Eq. (4) as

$$\phi \sim \exp\left[-\alpha y + \frac{ip}{R}y + \frac{i(q\alpha - R\,d\alpha/dx)}{2R}y^2\right] \tag{5}$$

In order that ϕ decay at large y, the real part of the exponent must be negative, *i.e.*,

$$2\alpha_r y + (q\alpha_i/R - d\alpha_i/dx)y^2 > 0. \tag{6}$$

Close to the neutral stability curve a; is small, so

$$\frac{d\alpha_i}{dx} = p\frac{d\alpha_i}{R} \gg \frac{q\alpha_i}{R}$$

and the second term in Eq. (6) can be neglected. At the lower branch of the neutral stability curve $d\alpha_i/dx < 0$, so ϕ always decays with increasing y. At the upper branch however $d\alpha_i/dx > 0$, but ϕ decays as y increases except for $y \sim 0(\alpha_r/\alpha_i)$. Since α_i is very close to 0 the real part of the exponent is negative upto a very large value of y, where the analysis is not valid since Eq. (4) has been derived assuming $y \sim o(R)$.

For the special case of Blasius flow, a comparison of the present formulation with those of previous workers is possible, and is given in the Appendix. For flows with pressure gradients there is no comparable formulation.

3. Numerical method

A code for the computation of the eigenvalues of the Orr-Sommerfeld equation by the Thomas algorithm was obtained from Sen (1989). This was modified to compute neutral stability curves for Falkner-Skan profiles by the non-parallel formulation described above. From Eq. (5) it is seen that outside the boundary layer the eigenfunction ϕ may be expressed as

$$\phi = f(x) \exp\left[-\alpha y + \frac{ip}{R} y + \frac{i\,(q\alpha - R d\alpha/dx)}{2R} y^2\right]$$

where $f(x)$ is fixed by the normalisation chosen for ϕ. The stability boundary of a physical quantity, however is independent of the normalisation. Here $f(x)$ is taken to be constant and therefore

$$\frac{d\alpha}{dx} = \frac{-i}{y\phi}\frac{\partial\phi}{\partial x} + o(R^{-1}). \tag{7}$$

$\partial\phi/\partial x$ is computed iteratively. It is found that the solution converges within two or three iterations.

4. Results and discussion

The results of the present formulation for a flat plate compare well with Gaster's non-parallel results as seen from Fig. 2. The experiments of Ross *et al.* (1970) measured the neutral location of the streamwise velocity fluctuations at about $y = 0.15\delta$, while Schubauer and Skramstad (1948) measured the same quantity at a constant (but unspecified) value of y_d below the inner maximum of the eigenfunction. The present results under similar conditions are shown in Figs. 3 and 4. In Fig. 4, the value of y_d is chosen so that it lies at about half the height of the inner maximum at the critical Reynolds number, *i.e.* the minimum Reynolds number at which instability occurs. It is seen that the discrepancy between experiment and linear stability theory is not reduced substantially by the inclusion of non-parallel effects. However, it is seen from these figures that stability is very sensitive to the location at which measurements are made. This fact has been pointed out by other workers too, *e.g.*, Fasel and Konzelmann (1990). This calls for experiments where the measured quantity and the path followed by the measuring probe are specified accurately.

In adverse pressure gradient flow, the reduction in critical Reynolds number due to non-parallel effects is significant, as seen from Fig. 5. The non-dimensional frequency plotted in this figure is given by $F = \omega/R^{(1-3m)/(1+m)}$. The effect of the non-parallel terms in the stability equation increases with increasing pressure gradient. Thus, a non-parallel stability analysis may be important for the understanding of the events leading upto transition to turbulence in such flows.

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Appendix

For flat plate flow, p=q and Eq. (4) reduces to

$$\mathcal{L}\phi + \frac{p}{R}\left\{\phi_0 D^3 + \phi_0' D^2 + \left[2y\alpha(\omega - \alpha\phi_0') - \alpha^2\phi_0 + \phi_0''\right] D + \left[-\alpha\omega + \phi_0'''\right]\right\}\phi$$

$$+\frac{d\alpha}{dx}\left(-\omega + 3\alpha\phi_0'\right)\phi + \left(\phi_0''' + 3\alpha^2\phi_0' - 2\alpha\omega - \phi_0' D^2\right)\frac{\partial\phi}{\partial x} = 0 \qquad (A1)$$

where £ is the Orr-Sommerfeld operator of Eq. (1). The above equation is rewritten as

$$\mathcal{L}\phi + \left\{a_3 D^3 + a_2 D^2 + a_1 D + a_0 + a_\alpha \frac{d\alpha}{dx}\right\}\phi + a_\phi \frac{\partial\phi}{\partial x} = 0 \qquad (A2)$$

where

$$\begin{aligned}
a_0 &= \frac{p}{R}\left[-\alpha\omega + \phi_0'''\right] \\
a_1 &= \frac{p}{R}\left[2y\alpha\left(w - \alpha\phi_0'\right) - \alpha^2\phi_0 + \phi_0''\right] \\
a_2 &= \frac{p\phi_0'}{R} \\
a_3 &- \frac{p\phi_0}{R} \\
a_\alpha &= -\omega + 3\alpha\phi_0' \\
a_\phi &= 3\alpha^2\phi_0' - 2\alpha\omega - \phi_0' D^2
\end{aligned}$$

The equation proposed by Sen (1992) is

$$\mathcal{L}\phi + \left\{a_3 D^3 + 2a_2 D^2 + a_1 D\right\}\phi = 0 \qquad (A3)$$

while the parabolic stability equation of Bertolotti *et al.* (1992) is given by

$$\mathcal{L}\phi + \left\{ a_3 D^{3} + a_2 D^2 + a_1 D + a_0 + a_\alpha \frac{d\alpha}{dx} \right\} \phi + a_\phi \frac{\partial \phi}{\partial x} = \frac{a_R}{R^2} \tag{A4}$$

where

$$a_R = 4i\alpha(D^2 - \alpha^2)(p - ypD + R\frac{\partial}{\partial x})\phi + 2i(R\frac{d\alpha}{dx} - \alpha p)(D^2 - 3\alpha^2)$$

Equation $(A4)$ is the same as Eq. $(A2)$ except for higher order terms, showing that the present formulation is consistent with that of Bertolotti *et al.* (1992) for Blasius flow upto $O(R^{-1})$.

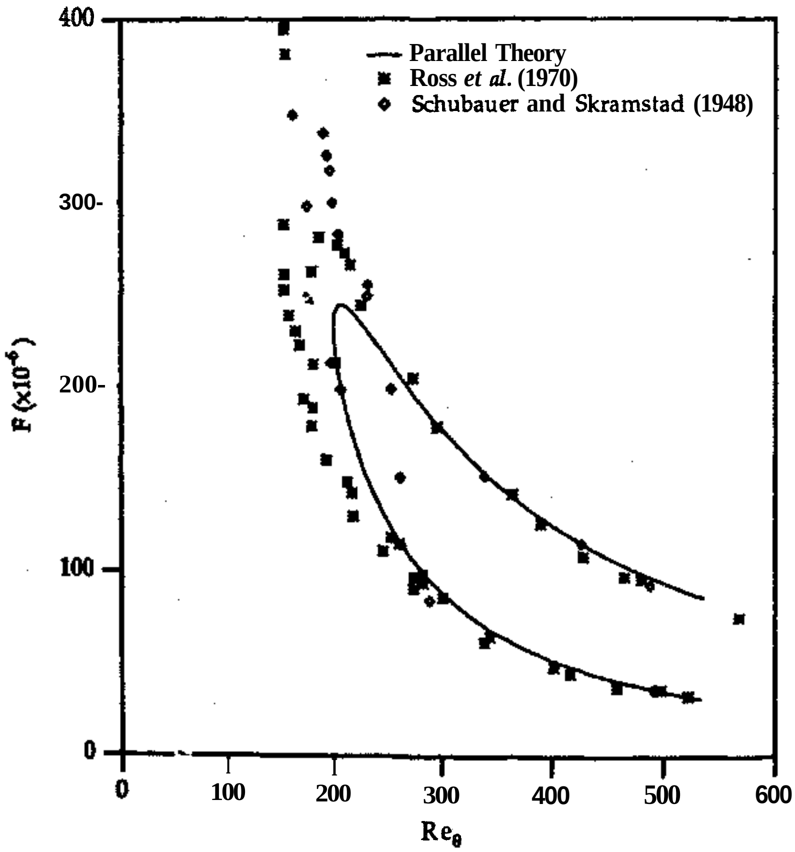


Figure 1. Comparison of parallel flow computations with experiment.

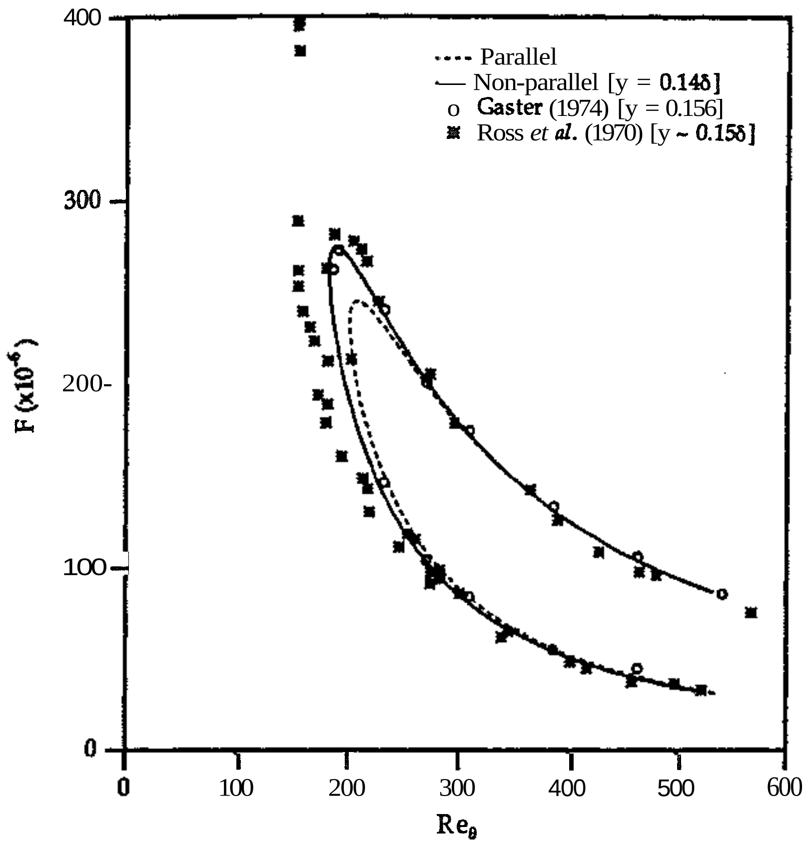


Figure 2. Results of the non-parallel formulation at constant y.

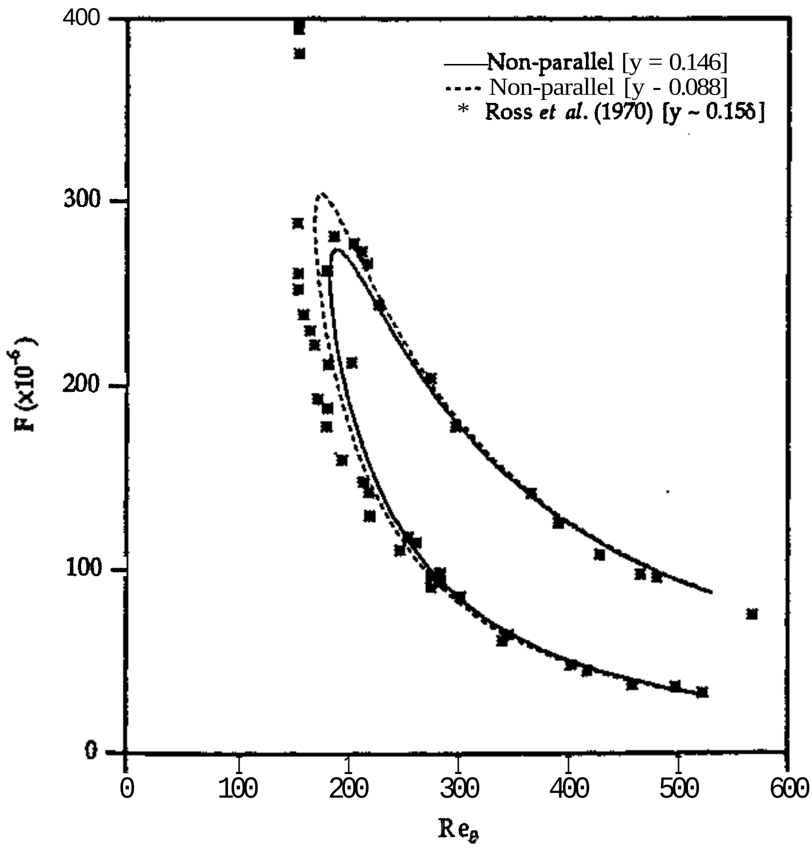


Figure 3. Results of the non-parallel formulation at constant y.

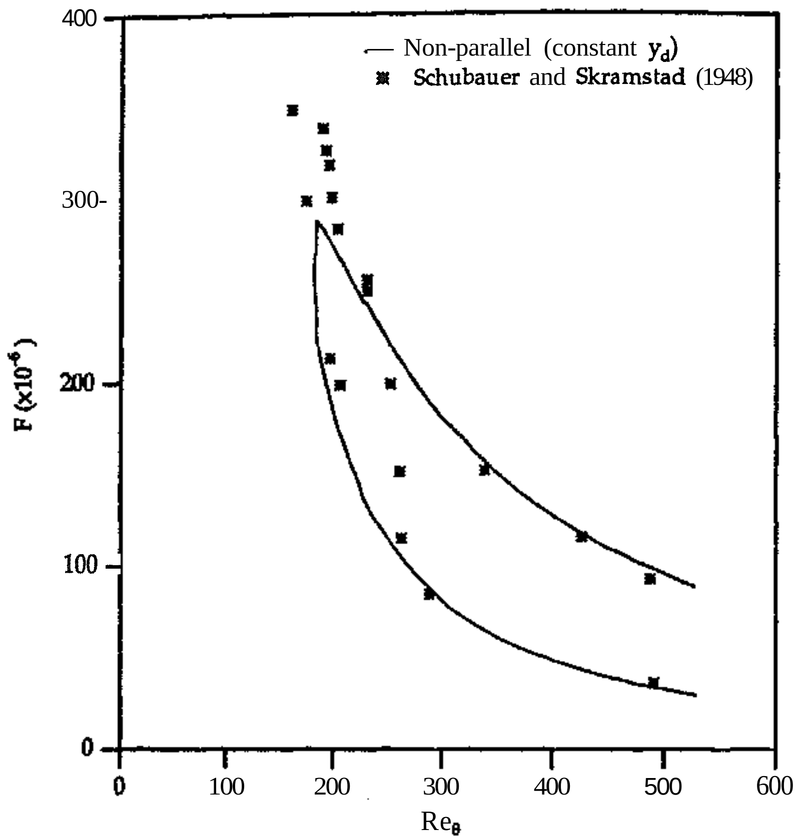
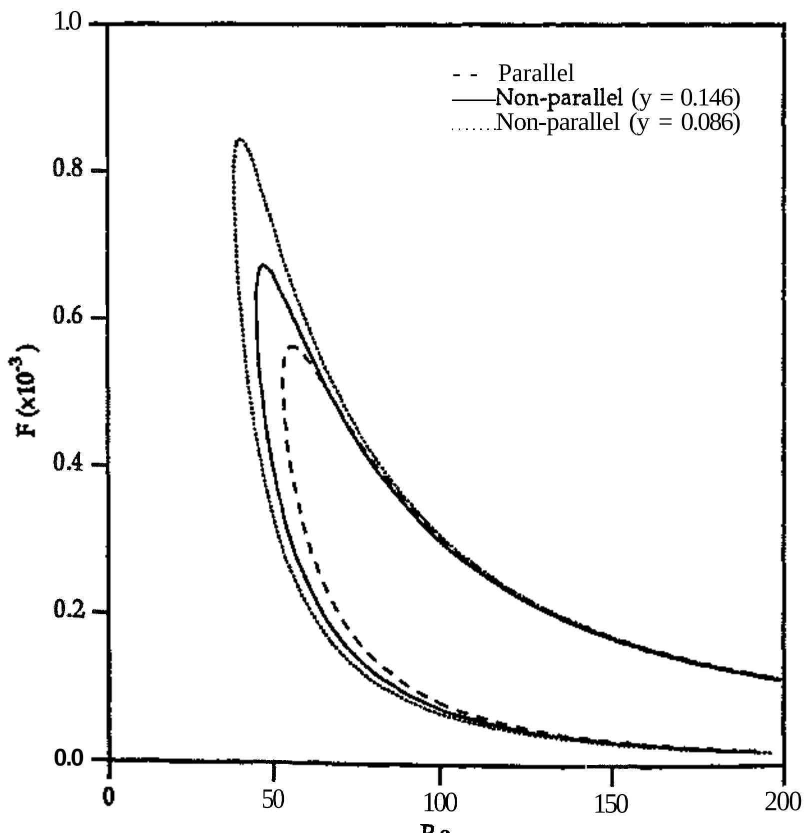


Figure 4. Results of the non-parallel formulation at constant y_d.



Rgure 5. Non-parallel effects in adverse pressure gradient flow (m=-0.06).